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Bifurcation theory: a tool for nonlinear flight dynamics

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THE ROYAL SOCIETY

Bifurcation theory: a tool for nonlinear flight dynamics

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This paper presents a survey of some applications of bifurcation theory in flight dynamics at ONERA. After describing basic nonlinear phenomena due to aerodynamics and gyroscopic torque, the theory is applied to a real combat aircraft, and its validation in flight tests is shown. Then, nonlinear problems connected with the introduction of control laws to stabilize unstable dynamic systems and transient motions are addressed. To extend the scope of applications, ongoing research devoted to the analysis of complex dynamic systems, including both continuous and discrete time parts, is mentioned. In conclusion, as a result of work undertaken at ONERA, it is stated that this theory is a useful tool for the study and control of high-dimensional dynamic systems.

Keywords: bifurcation theory; combat aircraft; dynamic systems; flight dynamics; flight tests; stability analysis

1. Introduction

It is now well known that the bifurcation theory can help predict the asymptotic behaviour of nonlinear differential equations depending on parameters. Efficient numerical procedures are now available and several previous studies have demonstrated that bifurcation analysis can be used to predict complex phenomena.

As in flight dynamics, aircraft motion is described by a set of nonlinear differential equations, depending on parameters, associating the state vector (angle of attack (AOA), sideslip angle, speed, angular rates, etc.) with the control vector (motivators, etc.) through motion equations, aerodynamic models, and flight-control systems. This paper aims to present results obtained in this field with a global stability analysis methodology making use of bifurcation theory.

After a brief presentation of the methodology and numerical procedures available, basic but very simple and well-known nonlinear phenomena, such as spiral mode, auto-rotational rolling, and Dutch-roll instability, are revisited by using bifurcation theory.

Then, the theory is applied to a real combat aircraft, the German–French Alpha-Jet from Dassault Aviation. After a brief description of the aircraft model, the oscillatory flight cases, such as 'agitated' spins, are studied by means of learning the stability characteristics of periodic orbits related to oscillatory unstable equilibrium points. Complex oscillatory modes are pointed out. The synthesis of all this illustrates that the lack of a realistic nonlinear model may lead to great difficulties for flight analysis when the motion is quasi-periodic or chaotic. Comparisons between predictions and flight tests at the French flight test centre are shown.

Control laws either stabilize unstable systems and/or increase their robustness under system modifications and perturbations. In practical situations, nonlinearities are numerous either in the dynamic system or in its control laws. Rather than detailing nonlinear control theory, it is shown that it can be very instructive to introduce bifurcation analysis while designing control, bearing in mind that the robustness of a stable steady state is closely related to its region of asymptotic stability.

From a theoretical point of view, parameter variations are assumed to be fixed and independent of time. When temporal parameter variations are not small, one can observe behaviour that is different from that initially predicted by means of bifurcation theory. In § 6 we discuss the connection between asymptotic behaviour and quasi-stationary and/or transient behaviour. These considerations are closely connected with the attracting-basin computation problem, which is also addressed.

The previous results have been obtained with continuous or 'almost-continuous' dynamic systems. To extend the scope of application in a more realistic way, it is necessary to be able to work with discrete-time systems and, finally, with complex dynamic systems including both a continuous part and a discrete-time part. Ongoing research devoted to the analysis of complex dynamic systems at ONERA is proposed in § 7.

2. Methodology

During the past 10–15 years, many methods have been suggested for the numerical solution of nonlinear problems (Guicheteau 1993*a*). This includes, in particular, the solution of parameter-dependent nonlinear equations by continuation techniques, and the related methods for bifurcation and stability analysis. Some of these codes deal with several aspects of the problem, while others concentrate only on specific aspects. All these codes are based on powerful continuation methods which are a direct result of the implicit-function theorem. However, it is noticed that continuation requires evaluation of the system and computation of partial derivatives, which can be very time consuming. Thus there is a need to improve the performance of such codes in order to get almost interactive procedures, even for high-dimensional systems. To this end, and for several years, an efficient numerical code, ASDOBI, for the analysis of continuous and discrete-time nonlinear-dynamic systems using robust continuation algorithms has been developed at ONERA (Guicheteau 1993*a*).

Once a numerical procedure is chosen, one has to set up a methodology to investigate the behaviour of the dynamic system.

The first step is to compute all the steady solutions of the system and their associated stability. As this step is generally time consuming, the computations have to be limited to the field of interest. Nevertheless, one must be very careful because the number of steady solutions for a given parameter is generally not known. Therefore, *a priori* qualitative experience on the global behaviour of the systems is preferable.

The second step consists of making graphic representations of the results in appropriate subspaces especially for high-dimensional systems. This step requires versatile graphic codes, and, again, a good experience of the system under consideration. Sometimes this step shows that equilibrium branches are missing.

The third step is concerned with the prediction of system behaviour when a bifurcation point is encountered. To achieve this, the computation of the attracting domain of the stable steady-states and, once more, the experience of the engineer, are very useful when investigating the various possibilities.

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Experience acquired by the processing of a large number of cases is extremely valuable for correctly predicting system behaviour. Thanks to this experience, it is not always necessary to check the predictions by means of numerical simulations. However, for the difficult cases, and if there is any doubt, the last step of the methodology consists of performing only a few numerical simulations before testing the predictions in simulation and on the real system.

3. Basic nonlinear phenomena

The prediction and analysis of control loss and spin of aircraft are old problems. However, with the lack of computer capabilities, previous techniques involved the application of exact or approximate analytical methods to simplified nonlinear equations, taking into account gyroscopic torque and some aerodynamic nonlinearities (Phillips 1948; Pinsker 1958; Hacker & Oprisiu 1974; Kalviste & Eller 1989; Ross & Beecham 1971; Padfield 1979; Adams 1978; Laburthe 1975).

The studies of Schy & Hannah (1977) and Schy *et al*. (1980) can be considered as two of the last works that can be related to a simplified treatment of control losses. They showed the possibility of multiple equilibrium solutions for a simplified set of flight equations, several of which are stable. These works formed the basis of the global methodology used at ONERA. These results have been revisited and completed by many scientists in the past two decades (see Guicheteau (1993*a*) and Littleboy & Smith (1997) and references therein). Nevertheless, most of the results presented in previous papers are similar, and exhibit basic nonlinear phenomena that are useful in gaining a better understanding of aircraft behaviour.

(*a*) *Spiral instability*

This slow motion occurs at low AOA when the aerodynamic model is symmetrical. Only gravity and pitch angle have an effect on the stability of the motion at zero sideslip angle.

When lateral control deflections are at neutral, the equilibrium surface of the system versus elevator deflection shows that spiral instability is bounded by two fork bifurcations on the lateral variables of the system (Guicheteau 1993*a*). Between these two limits of stability, the aircraft is unstable in straight level flight and, in response to a lateral disturbance, it tends towards a turning-down flight, the characteristics of which are determined by the stable equilibrium branches located between the two limits of stability at zero sideslip angle (figure 1).

By comparison with the classical linearized flight dynamics, this methodology is able to predict system behaviour beyond the limit of stability. It also gives an idea of the nonlinearities responsible for the instability. Thus, considering the essential nonlinearities, it is possible to analyse aircraft behaviour by reducing the motion equations to a scalar equation relating roll angle, pitch angle, and control deflections:

$$\dot{\Phi} = (A \sin \Theta + B \cos \Theta \cos \Phi) \sin \Phi + (C_{\delta a}\delta a + C_{\delta r}\delta r) \cos \Phi + D_{\delta a}\delta a + D_{\delta r}\delta r,$$

in which A, B, C and D are coefficients that depend on the aerodynamic characteristics. In particular, B is the classical stability criterion for the linearized motion equations.

Despite its simplicity, this formulation synthesizes aircraft behaviour in the vicinity of the spiral instability. Moreover, it shows that spiral bifurcation can appear in the aileron–rudder plane even if level flight at zero sideslip angle is stable (figure 2).

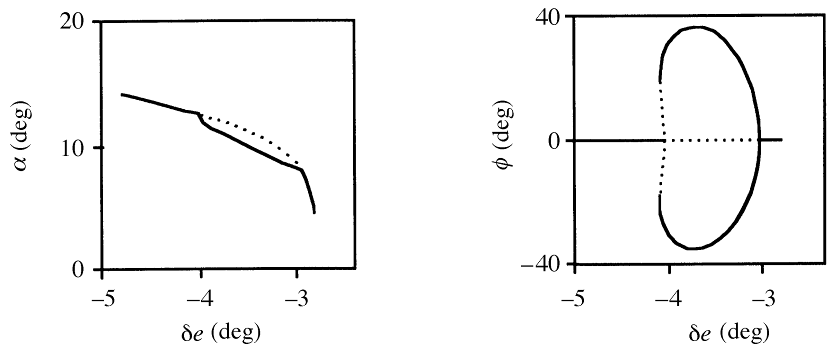

Figure 1. Spiral bifurcation: ——, stable; – – –, unstable divergent.

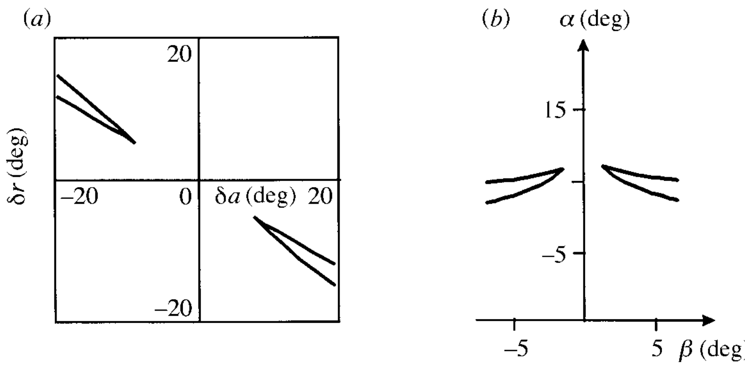

Figure 2. Spiral bifurcation in aileron–rudder plane.

(b) *Auto-rotational rolling*

Experiments and previous computations have shown that auto-rotational rolling occurs at low AOA. It can also be seen that speed varies only a little and that the influence of gravity is negligible. Then, assuming also that pitch rate and yaw rate are much smaller than the roll rate, it is still possible to transform the analysis of the motion equations into the study of a polynomial nonlinear scalar equation, similar to the canonical form of a singularity, which is called a 'butterfly catastrophe', relating yaw rate and control deflections:

$$\dot{p} = f_6(\delta_e)p^5 + (f_{5_{\delta a}}\delta a + f_{5_{\delta r}}\delta r)p^4 + f_4(\delta_e)p^3 + (f_{3_{\delta a}}\delta a + f_{3_{\delta r}}\delta r)p^2 + f_2(\delta_e)p + (f_{1_{\delta a}}\delta a + f_{1_{\delta r}}\delta r),$$

in which the coefficients f_i depend on aerodynamic characteristics and gyroscopic torques (figures 3 and 4).

Taking into account gravity and speed effects, the real behaviour of the aircraft shows that gravity effects result in an oscillation of the state variables around a mean value, and cause the aircraft to dive and accelerate. It follows that the roll rate does not stabilize at the predicted value. More precisely, it is observed that it is possible to rewrite the system under study by using the reduced roll rate and verify that, despite the increase in aircraft speed and roll rate, the value of the reduced roll rate is perfectly stabilized at the value anticipated by the equilibrium computations. The reduced angular rate is therefore a good indicator of the behaviour.

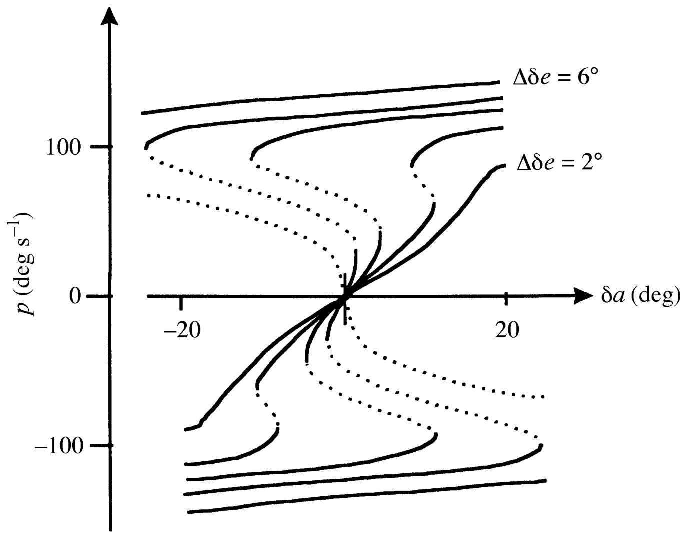

Figure 3. Inertia coupling: steady roll rate versus aileron deflections: ——, stable; – – –, unstable divergent.

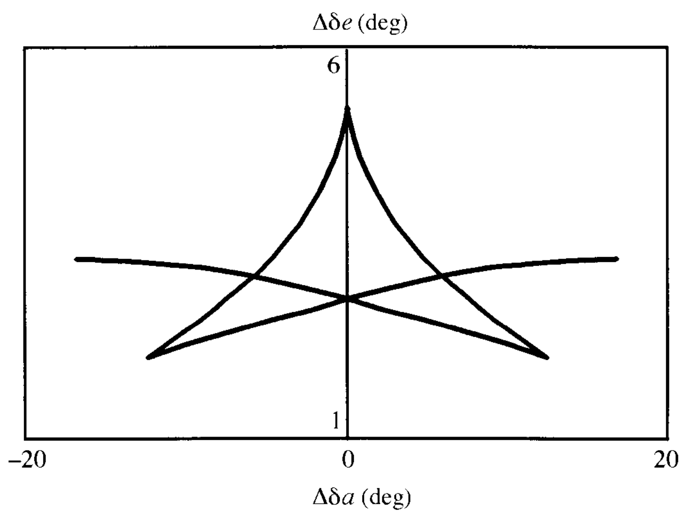

Figure 4. Inertia coupling: bifurcation surface in aileron–elevator plane.

Moreover, once reduced to a scalar equation, a control system which avoids bifurcations can be easily designed. In practice, such a system is an aileron–rudder coupling, which is used on many aircraft. Although it does not modify the pattern of the bifurcation curves, that is intrinsic to the aircraft, it does modify their occurrence; the possible equilibria are no longer the same, and are not so varied.

(*c*) *Dutch-roll instability*

This phenomenon is very well reported in the literature (Guicheteau 1993*a*). In this case, instability is connected to a Hopf bifurcation point, which is approximated

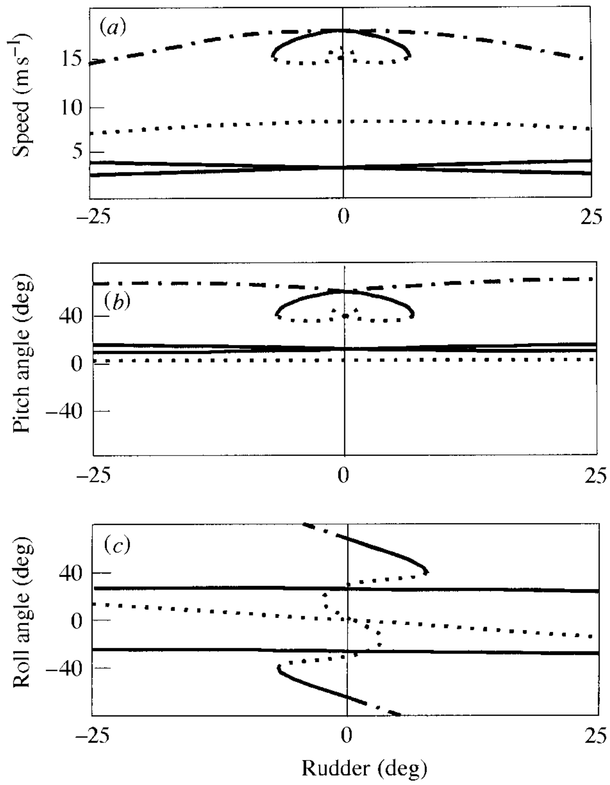

Figure 5. Equilibria of a submarine model. ——, Stable; – – –, unstable divergent; –·–·–, oscillatory unstable.

by classical theoretical and experimental handling quality criteria ($c_{n_\beta \mathrm{dyn}}$, Kalviste, etc.).

In this case, the first point of interest of bifurcation theory is the characterization of the Hopf bifurcation (subcritical or supercritical) (Guicheteau 1986) in order to get an indication of the evolution of the amplitude of the periodic motion beyond the limit of stability. Then it is of interest to compute the periodic-orbits envelope, without the usual simplified assumptions, in order to investigate secondary bifurcations which can take the appearance of wing rock or spin.

(d) *More complex phenomena*

So far, it has been shown that bifurcation theory is a powerful tool for understanding several classical nonlinear flight-dynamics phenomena, for which a linearized approach is not suitable. Similar phenomena have been also exhibited at ONERA with models of submarines for which the aerodynamic part is replaced by an hydrodynamic one. As an illustration, figure 5 presents the computed equilibria of a submarine model during an emergency manoeuvre with degraded control (Pavaut 1993). Several of them are very similar to the inertia coupling and spin of aircraft. However, as on combat aircraft, more complex phenomena are also encountered.

Computation of periodic-orbit envelopes and their bifurcations shows that much agitated behaviour can also be analysed by means of bifurcation theory. In particular, when two conjugate imaginary eigenvalues of the transition matrix of a periodic orbit cross the unit circle while a parameter varies, the initial stable orbit becomes unstable and the motion lies on a toroidal surface surrounding the newly unstable orbit. Generally, in this particular case, the motion seems to be a superposition of two periodic motions with very different periods. Finally, when all the equilibrium states are unstable except for one which is weakly stable, very long transient motion can appear. Sometimes, this motion seems to be complex and/or chaotic. So this could result in some difficulties when analysing and modelling such phenomena from flight tests because the running time is generally less than the larger period. More precisely, and in case of sensitivity to initial conditions, if the identification process ignores the existence of such motion, and the influence of typical nonlinearities, it then results in an identified model that matches only the flight-test data used in the identification process. It will not exhibit the sensitivity to initial conditions.

4. Application to a real combat aircraft

The Alpha-Jet is a tandem two-seat German–French aircraft for close support and battlefield reconnaissance. With narrow strake on each side of the nose, it is also an advanced jet trainer. Considering its great ability to safely demonstrate numerous and various high-AOA behaviours and for flight-test correlation, the training version was chosen to investigate the interest of the methodology (Guicheteau 1993*b*).

(*a*) *Aircraft model*

Each of six global aerodynamic coefficients is expressed independently as a function of flight and control parameters. A general expression of coefficients can be expressed in the form,

$$c_i = c_{i\ \mathrm{stat}} + c_{i\ \mathrm{unstat}},$$

where the coefficients $c_{i\ \mathrm{stat}}$ and $c_{i\ \mathrm{stat}}$ represent stationary aerodynamic effects and take into account unsteady effects expressed as a transfer function, respectively. All these terms have been measured and tabulated over a wide state and control domain.

Generally, in order to prevent continuation problems, aerodynamic coefficients are usually smoothed to ensure continuity and derivability conditions for the resulting nonlinear dynamic system. In our application, no preliminary smoothing was done to avoid pure numerical bifurcation points and behaviour. Coefficients are evaluated by linear interpolation of the tabulated data.

(*b*) *Predictions and flight tests*

The results presented here are related to control losses, spin and spin recovery. In order to simplify the interpretation of computations, only typical cases will be shown. Before discussing the results, it should be noticed that they are extracted from classified studies. So the angular rates are plotted without a scale.

Starting from a straight level flight at low AOA, when the pilot moves the elevator for a full nose-up attitude, we can observe multiple steady-states appearing at high AOA when both aileron and rudder deflection vary. The projection of the equilibrium surface in characteristic subspaces easily allows identification of the domains of spin

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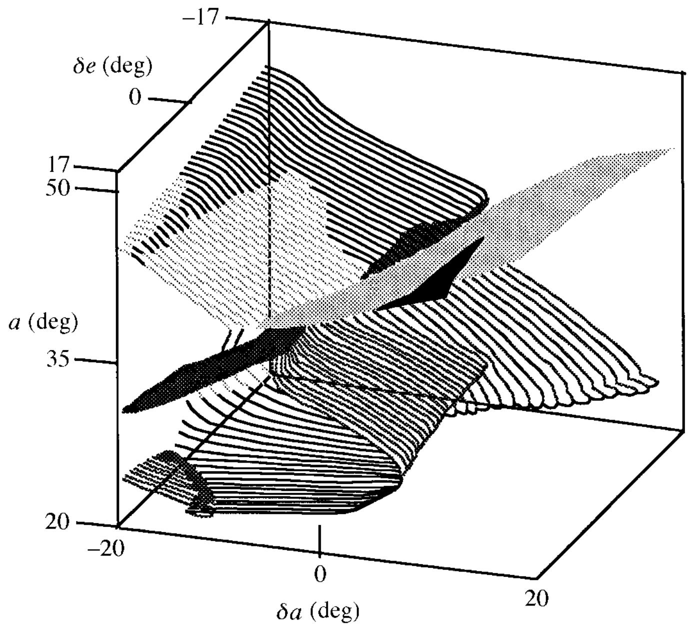

Figure 6. Equilibrium surface projection in a characteristic subspace (aileron, rudder, incidence).

and rolling motions (figure 6). It should be observed that this surface is not symmetrical. This is due to non-symmetrical aerodynamic data for symmetrical aileron and rudder deflections at high AOA.

As can be seen in figure 6, stability is very different from one point to another. More precisely, it seems that left spins, related to negative aileron deflections, are much more unstable than right spins. Perhaps this low degree of stability can explain pilot difficulties in demonstrating left steady spins on an Alpha-Jet. The third type of steady-state encountered on this surface, corresponds to an important roll motion at moderate AOA. In flight, this kind of motion occurs mainly when pilots fail spin entry or fail the transition from one spin on one side to another spin on the other side.

Considering the equilibrium curve for full rudder deflection (figure 7), it can be seen that right spin is stable while left spin is always oscillatory unstable except for a few positive aileron deflections.

Surrounding this last equilibrium branch for negative aileron deflections, there exist several periodic orbits when aileron deflection decreases. The amplitude of AOA and roll rate of the computed orbits are plotted versus aileron deflections in figure 8. Two distinct branches can be observed. On the first one, the limit points are numerous. For small deflections, two convergent series of flip periodic bifurcations determine a region in which an Alpha-Jet can exhibit chaotic behaviour. In our case, contrary to typical chaotic behaviour exhibited by well-known particular differential equations, there are only small differences in amplitude between the different orbits of period T, $2T$, etc. It seems then, that this behaviour will be very difficult to observe and to characterize in flight. Finally, the most important phenomena on this branch

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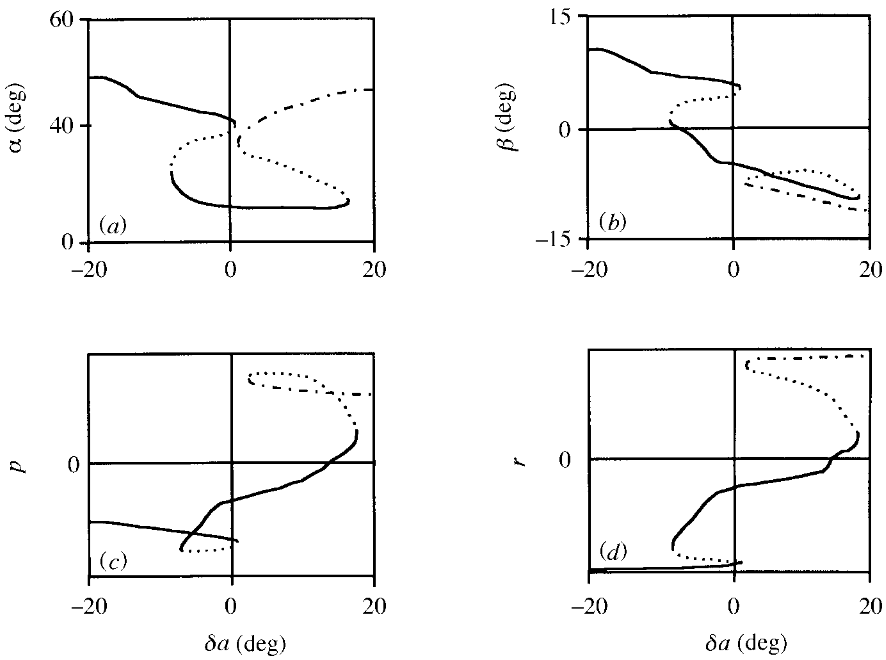

Figure 7. Equilibrium curve for $\delta_r = 17°$: ——, stable; · · · · · ·, unstable divergent; –·–·–, oscillatory unstable.

of the envelope is the rapid variation of orbit amplitude with aileron deflection. This could explain the level of agitation, which is well known to pilots.

On another branch of the periodic envelope, and for deflections from −10 to −20°, there exist oscillatory unstable orbits with great amplitude (figure 8). Around them, the motion takes place on a toroidal surface if it is stable. Nevertheless, the existence, for some deflection of an invariant torus and a stable orbit, can lead to non-similar flight behaviour. This different behaviour depends on the initial state and on the history of control deflections during the manoeuvre.

All these phenomena have been demonstrated in simulation. In order to make a correlation between predictions and flight, flight tests have been done at the French Flight Test Centre in Istres. Before presenting a few of the results obtained during the flight tests, one must bear in mind that good correlation indicates only that the aircraft model is realistic. It is not a validation of the methodology which had been already validated through numerical simulations.

For full nose-up elevator and positive rudder deflections, quiet left spin is obtained for a small aileron deflection (figure 9). When aileron deflection is close to −10°, the Alpha-Jet can exhibit three very different motions due to the existence of two stable orbits and a stable invariant torus (figures 10 to 11).

Finally, it can be noticed that chaotic motion for small aileron deflections was not demonstrated. This result is due to the short duration of spin tests and to the absence of great differences between the orbits in presence, as previously mentioned. Nevertheless, this phenomenon was known by the pilots who experienced it, and who knew that it was impossible to achieve similar flight tests with such deflections.

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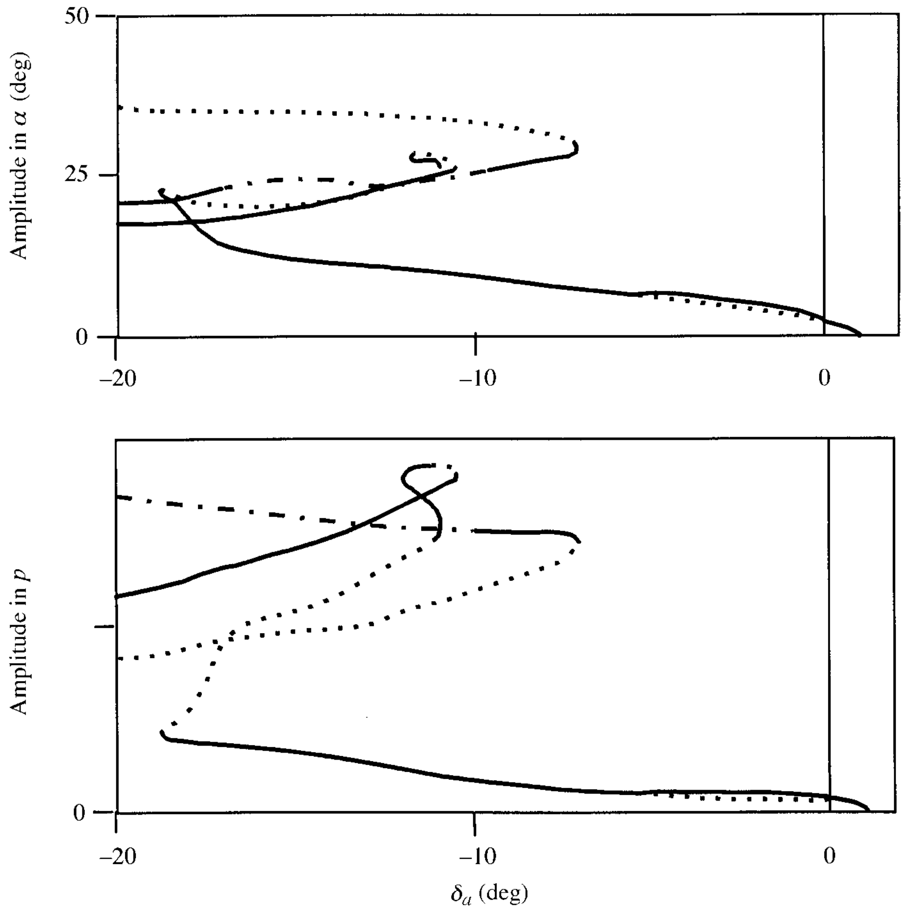

Figure 8. Amplitude of periodic orbits when δ_a varies between −20 and −17°: ——, stable; · · · · · ·, unstable divergent; —·—·—, oscillatory unstable.

5. Some remarks about control problems

As opposed to many theoretical approaches, practical control laws are generally nonlinear because of their formulation or the use of nonlinear elements. Thus, it is interesting to investigate whether the proposed methodology can help the designer to design 'good' control laws from a stability point of view. More precisely, one must answer the following question: can 'stabilizing' control laws introduce new bifurcations while they are used to 'stabilize' the open-loop system?

Without invoking bifurcation theory, these control problems are being studied and, fortunately, 'good' solutions have been found in many particular cases. A considerable number of results are already available in the literature.

In this section we are concerned with an autonomous dynamic system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)),$$

where $\boldsymbol{x}$ and $\boldsymbol{u}$ denote state and control vectors, respectively. $\boldsymbol{f}$ and $\boldsymbol{x}$ are n-dimensional vectors, $\boldsymbol{u}$ is an m-dimensional vector and $\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ are nonlinear functions satisfying Lipschitz conditions in which the control vector depends on the

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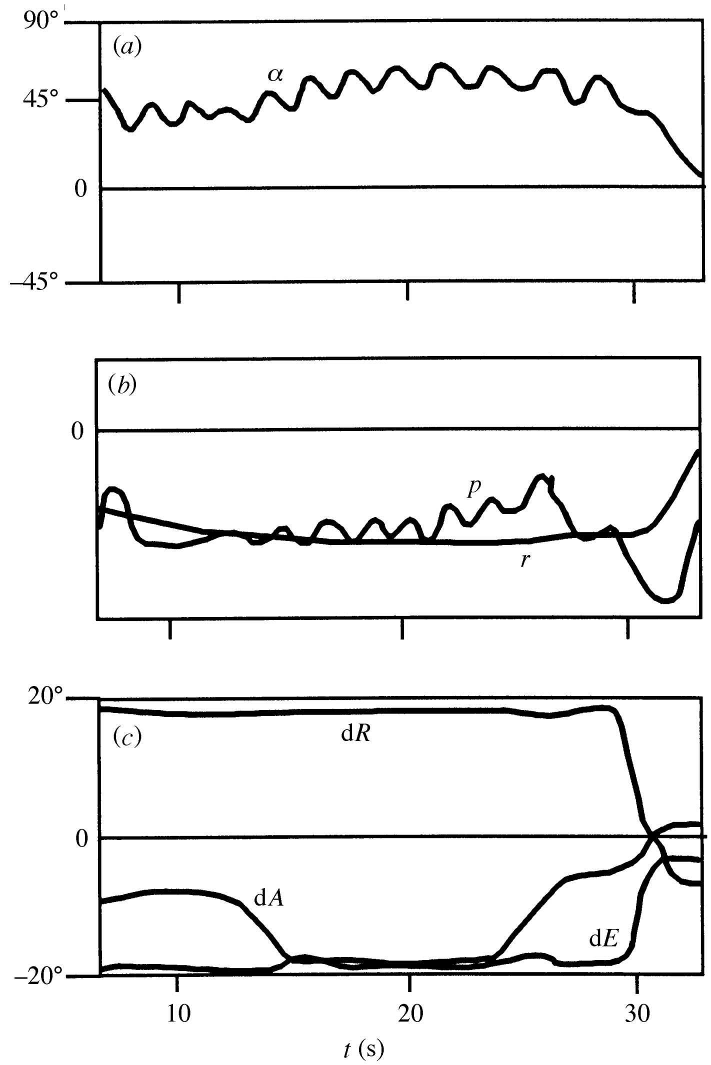

Figure 9. Quiet left spin for $\delta_a = -4°$.

state variables in the following way:

$$\boldsymbol{u}(t) = \boldsymbol{g}(\boldsymbol{x}(t), p),$$

where p represents new control parameters and $\boldsymbol{g}(\boldsymbol{x}(t), p)$ are nonlinear functions satisfying Lipschitz conditions. From the definition of the control vector, it follows that every equilibrium point of the open-loop system is also an equilibrium point of the closed-loop system and vice versa. However, the stability of the closed-loop solutions can differ from the stability of the open-loop solutions and new complex asymptotic solutions may appear.

As a first example, let an unstable second-order linear differential equation be stabilized by a control law which depends on a nonlinear element as described in figure 12.

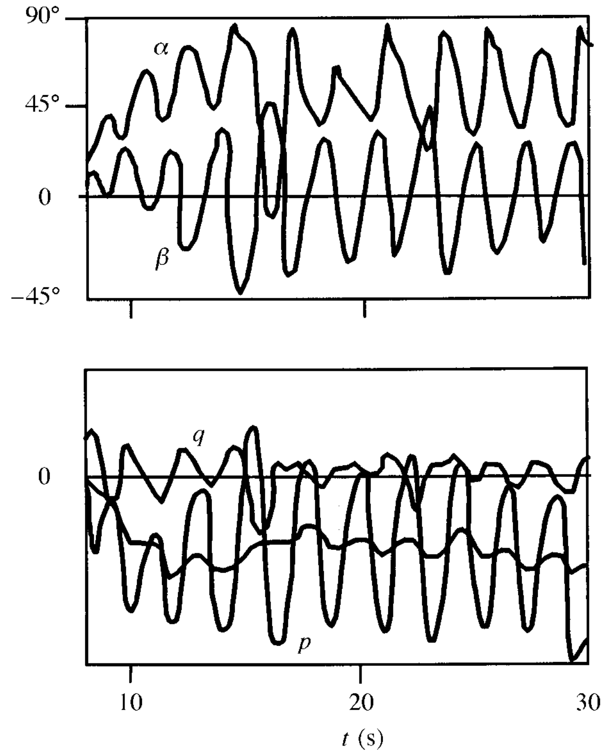

Figure 10. Regular agitated spin for $\delta_a = -10°$.

A linearized analysis of the closed-loop system,

$$\ddot{x} + \left(2\zeta_0\omega_0 + \frac{\partial k(\dot{x})}{\partial \dot{x}}\right)\dot{x} + \omega_0^2 x = e,$$

shows how gain influences the stability of the asymptotic solution. When $(\partial k(\dot{x})/\partial \dot{x})$ goes from zero to the adopted value, k^*, for the closed-loop system, it crosses a value k^c for which the steady-state admits two conjugate pure imaginary eigenvalues.

From a nonlinear point of view, one can say that there is a Hopf bifurcation in k^c. Moreover, consider a nonlinear gain,

$$k(\dot{x}) = \frac{2}{\pi} k(\dot{x})_{\text{lim}} \arctan \frac{k\pi\dot{x}}{2k(\dot{x})_{\text{lim}}},$$

i.e. the feedback is almost linear ($k(\dot{x}) \cong k$) in the vicinity of the equilibrium points and is saturated ($k(\dot{x}) \cong k(\dot{x})_{\text{lim}}$) when $\dot{x}$ tends to infinity. Then, one can easily show that the Hopf bifurcation is subcritical. Thus, when k is greater than k^c, periodic orbits surround the stabilized equilibrium points. These orbits can exist even for the adopted gain value k^* (figure 13).

It can be seen, in this two-dimensional case, that the unstable periodic-orbit envelope visualizes the boundary of the attracting region of the controlled system. As an example, if the amplitude of a perturbation is greater then the amplitude of the periodic orbit, the controlled system exhibits a divergence.

Applied to a typical combat aircraft with unstable lateral modes that is stabilized by a continuous feedback with a saturation on angular rates and stops on

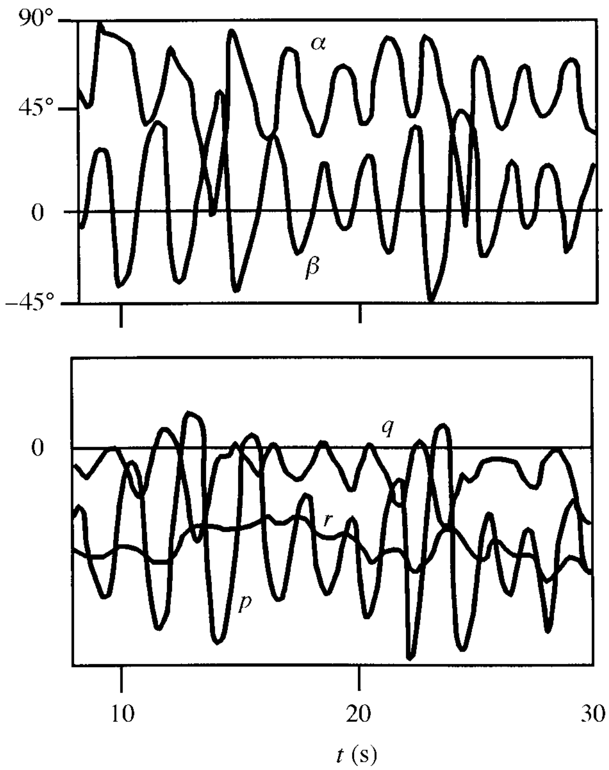

Figure 11. Motion on a toroidal surface for $\delta_a = -10°$.

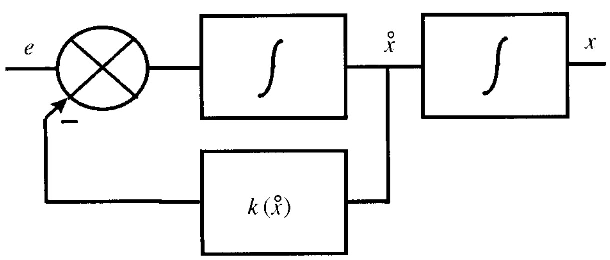

Figure 12. Controlled unstable second-order dynamic system.

lateral control deflections, computation of the unstable orbit surrounding the stabilized equilibrium point gives a first insight into the 'robustness' of the control law (figure 14). Moreover, stops on control deflections generate a stable periodic orbit which surrounds the unstable limit cycle and limits the divergence of the motion. As the dimension of the system is greater than two, the unstable periodic orbit is only one element of the attracting-region boundary.

Similar classified results had been obtained at ONERA when this methodology was applied to a modern combat aircraft and to a realistic model of an air-to-air missile for which the dimension of the system is greater than 50. Except for the difficulty related to the dimension of the system, the true difficulty was related to the numerical procedure because the real nonlinear elements were not C^{∞} functions (saturation,

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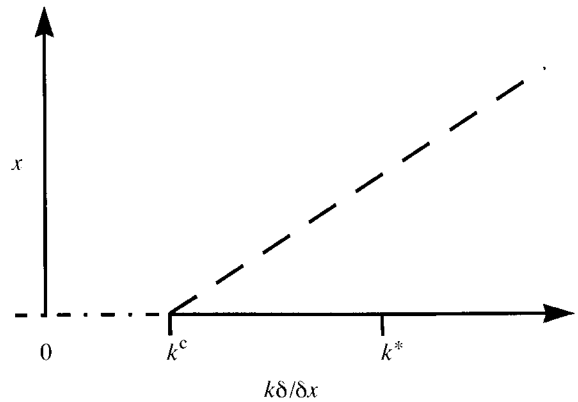

Figure 13. Amplitude of the steady solutions as a function of the nonlinear control gain: ——, stable; – – –, unstable; –·–·–, oscillatory unstable.

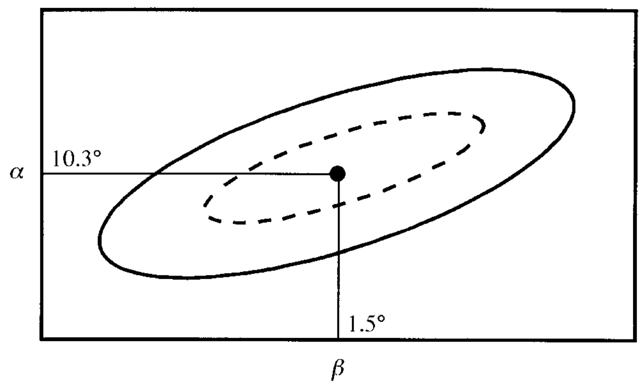

Figure 14. Fixed point and periodic orbits for a typical stabilized combat aircraft. ——, stable orbit; – – –, unstable orbit.

stop, hysterisis, etc.). So, it was necessary to improve the continuation algorithm to work with such singularities. The first improvement consists of predicting a new equilibrium solution form the previous one by using a nonlinear extrapolation based on non-equidistant points. This makes the continuation process easier when the equilibrium curve is highly nonlinear. The second major improvement is the ability of the continuation process to detect when the system is independent of a variable or a parameter, to automatically reduce the dimension of the system and to continue the computation with the reduced system. This case is generally encountered when controlled systems contain hysteresis effects and saturation on actuators. However, a lot of work remains to be done in order to characterize the stability of such systems.

Now, many control laws are provided by a computer with a sampling period. In order to increase the interest of the methodology for practical applications, it is necessary to be able to investigate the behaviour of complex dynamic systems which are composed of both a continuous part (motion equation) and a discrete time part (control law). In particular, one needs to characterize the stability of perturbed equilibria between sampling times. After solving the analysis of discrete-time systems, ONERA studies are now investigating numerical procedures to analyse these complex dynamic systems and new results will be available soon.

6. Attracting-basin and transient behaviour

A stable equilibrium state of a nonlinear-dynamic system is surrounded by a stability region. The determination of this region is of great interest for dynamicists and engineers. It allows definition of the limit of validity of linearized approximations for the original nonlinear equations, better understanding of the global behaviour of the system, and determination of the maximum values of the perturbations for which the perturbed system returns to the initial stable state.

The aim of this section is to present several methods which are currently in use, in order to give an answer to the attracting-basin computation problem for sets of ordinary nonlinear differential equations.

(*a*) *Preliminaries*

The systems under consideration are of the general form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)),$$

where $\boldsymbol{f}$ and $\boldsymbol{x}$ are n-dimensional vectors, and where $\boldsymbol{f}(\boldsymbol{x}(t))$ are nonlinear functions satisfying Lipschitz conditions. It exhibits, at least, a steady state x^*

$$\frac{\mathrm{d}\boldsymbol{x}^*(t)}{\mathrm{d}t} = 0, \qquad t > t_0,$$

or a periodic orbit defined by

$$\dot{\boldsymbol{\Phi}}^*(t) = \boldsymbol{f}(\boldsymbol{\Phi}^*(t)),$$

where

$$\boldsymbol{\Phi}^*(t+\tau) = \boldsymbol{\Phi}^*(t), \qquad \tau = NT,$$

where N is a positive integer and T the period.

Considering a periodic orbit, and without loss of generality, we can assume that $x^* = 0$ and that $t_0 = 0$ and notice that by defining $\xi = x - \Phi^*(t)$, the study of periodic solutions for a time-invariant system is transformed to the study of periodic systems (Poincaré map).

The domain of attraction (attracting basin, region of asymptotic stability) of a steady-state or of a periodic orbit, is defined as the set of all initial conditions, $\boldsymbol{x}_0(t_0)$, that tend, respectively, to x^* or to $\Phi^*(t)$ when time tends to infinity.

Numerous methods have been proposed in the literature for estimating the region of asymptotic stability. They may be roughly divided into Liapounov and non-Liapounov methods. Nevertheless, the distinction between these different classes of methods is less obvious now because modern methods generally use joint approaches.

(*b*) *Liapounov methods*

The methods using Liapounov functions are derived from the results obtained by Liapounov. Two approaches have been developed by using either results from Zubov or an extension of Liapounov's theorems produced by La Salle.

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The first approach may be applied to low-dimensional nonlinear systems with an exactly defined structure, and, generally, give conservative results. For higher-dimensional systems, several optimization approaches are used to modify an initial Liapounov function in order to enlarge the volume of the attracting region (Michel *et al*. 1982).

The second approach is related to the application of the concept of absolute stability in the frequency domain, as proposed by the Popov criterion, choosing a suitable Liapounov function holding for a whole class of nonlinear systems defined by a sector condition in the sense of Aizerman. In this case, the results obtained with this approach are specific to a type of nonlinear system but they are more general than the previous one. Nevertheless, they are also too conservative.

(*c*) *Trajectory-reversing method*

This method is known as the trajectory-reversing method or backward mapping. It is based on the La Salle extension of the Liapounov stability theory. It provides an iterative procedure for obtaining the global attracting region for multi-dimensional systems, both time-invariant and time varying without conditions on the topological nature of the asymptotically stable point under study.

Once an initial estimation of the attracting region bounded by the curve c_0 is made, a curve C_j is obtained by backward integration in time of the dynamic equations from $t = 0$ to $t = t_j$. Then, if $c_{-\infty}$ denotes the map of the curve c_0 as $t \to -\infty$, due to the uniqueness of the solution of the system, $c_{-\infty}$ is the domain of attraction of the origin. Generally, it is not necessary to compute $c_{-\infty}$ to get a good approximation of the true domain of attraction.

The trajectory methods are attractive because of their generality and simple theoretical framework. However, their computational efficiency is generally poor and they have only been used for low-dimensional systems.

To reduce the computational effort, $(n-1)$ facets can be used to approximate the basin boundary of an nth order system (Piaski & Luh 1990). Starting from a local quadratic Liapounov function around the stable equilibrium point under study, a small convex polytope† is generated. Then the vertices of this initial polytope are integrated backward in time to generate the vertices of a non-convex polytope approximation of the basin boundary. Thus, the real image approximates the attracting region as backwards integration time approaches infinity. However, as the system is a nonlinear one, a test is applied to check whether the new non-convex polytope is a good approximation of the image of the original convex polytope. Adaptative facet refinement is used to correct any inaccuracy of the image approximation. Even when applied to low-dimensional systems, a great number of vertices are still required for accurate approximation. More generally, extension to higher-dimensional cases is actually limited by the rapid growth of the (facet number)/(vertex number) ratio with the growth of the state-space dimension.

(*d*) *Differential geometry method*

This method gives a complete characterization of the stability boundary for a fairly large class of nonlinear autonomous dynamic systems satisfying two generic properties plus one additional condition that every trajectory on the stability boundary

† A polytope is a finite, flat-sided solid in any high-dimensional space.

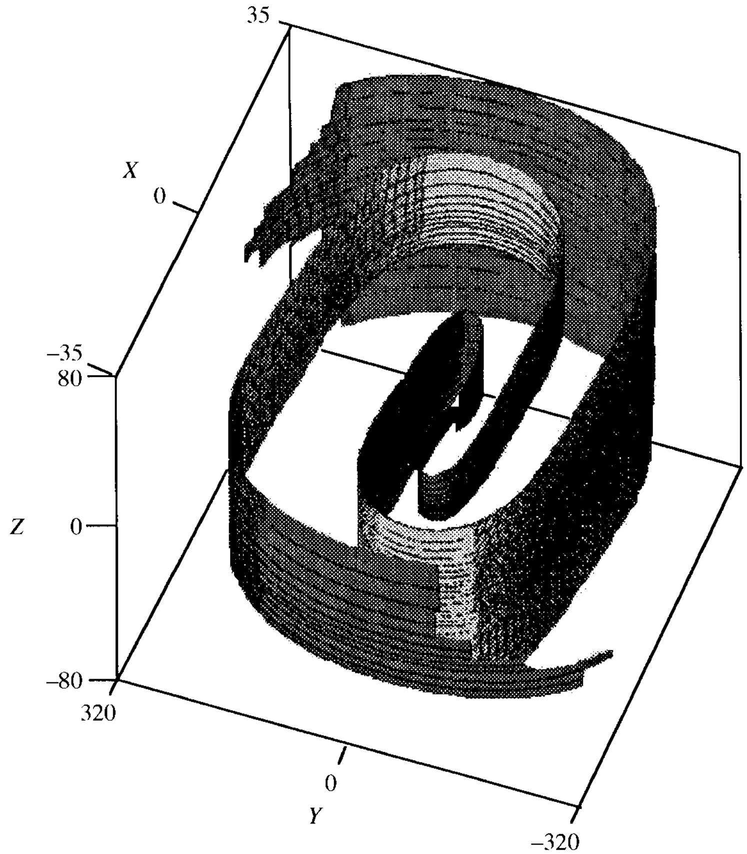

Figure 15. Partial view of the boundary of the domain of stability.

approaches one of the equilibrium states (fixed points or/and periodic orbit) as the time t tends to infinity. Then, it is shown that the stability boundary of this class of nonlinear systems consists of the union of the stable manifolds of all equilibrium states on the stability boundary (Chiang *et al.* 1988). With all these results, a numerical procedure can be set up to determine the stability boundary by means of the construction of the stable manifolds of all the equilibria which belong to it.

As an example, after computing the two stable equilibrium points, $(0,0,0)$ and $(-7.45,-7.45,-7.45)$, and the unstable equilibrium point $(-2.45,-2.45,-2.45)$ of the following system:

$$\dot{x} = -x + y,$$
$$\dot{y} = 0.1x + 2y - x^2 - 0.1x^3,$$
$$\dot{z} = -y + z,$$

the procedure enables us to compute the stability boundary of the attracting domain of the stable equilibrium points as the stable manifold of the unstable equilibrium point. A partial view of it is shown in figure 15.

For higher-dimensional systems, graphic representation of the boundary is difficult. Nevertheless, their projection in particular subspaces gives useful information. Figure 16 shows a partial view of the stability boundary between two stable equi-

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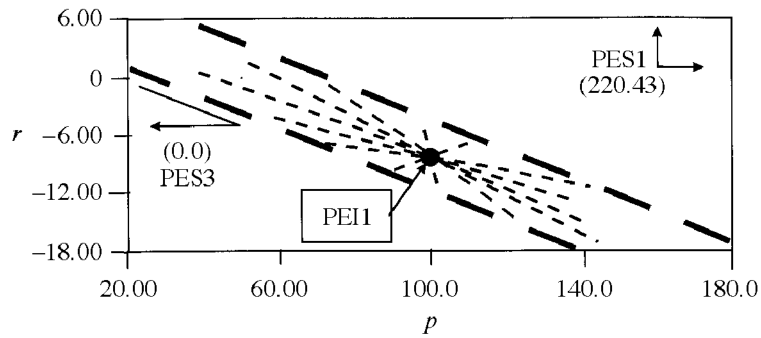

Figure 16. Projection in the (roll rate, yaw rate) plane of the boundary of the domain of stability. PES1 and PES2 are stable equilibrium points while PEI1 is an unstable one.

librium points of a set of five nonlinear equations describing aircraft motion at low AOA under inertia-coupling conditions.

(*e*) *Transient behaviour*

In many studies, the value of the parameter is considered to be fixed and independent of time. In many practical situations, this is not the case, and the system exhibits quasi-stationary behaviour and transient motions; the difference between these two behaviours is the value of parameter change over time. If it is slow in comparison to changes of state variables, quasi-stationary behaviour is observable. There are at least two situations to be considered:

(1) movement along a stable branch;

(2) movement through bifurcation points.

In the first situation, it has been shown that if the system has a stable manifold and fixed points corresponding to constant inputs, then an initial state close to this manifold and a slowly varying input signal, in an average sense, produce a trajectory that remains close to the manifold.

The second situation leads to different behaviour regarding the nature of the bifurcation point encountered. As an illustration, figure 17 shows possible situations.

Case (a) corresponds to a simple bifurcation point. A solution continues after the bifurcation point along a stable branch. In case (b), where the bifurcation is also a limit point, the branch on which the solution continues is chosen at random. In practical applications, the behaviour of the system has to be formulated statistically; the character of distribution of fluctuations of state variables determines the probability of the choice of individual branches of solutions. Case (c) is a very interesting one because after crossing the limit point the system evolves into the closest stable state, i.e. a state in whose domain of attraction the limit point belongs. The last case, (d), corresponds to the Hopf bifurcation. Generally, the apparition of the stable periodic orbit seems to be delayed and the low-amplitude solution around the bifurcation point is unobservable (Neishtadt 1987).

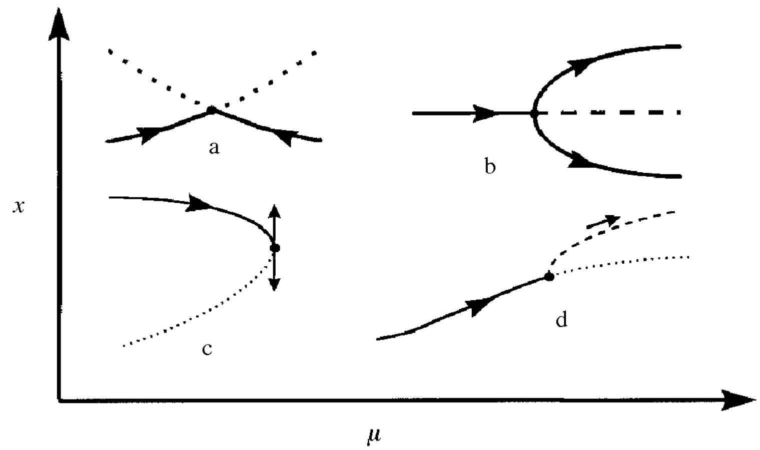

Figure 17. Examples of possible crossing bifurcation and limit points during quasi-stationary behaviour.

7. Conclusion

The behaviour of a fighter aircraft at AOA flight is so complex that it is very difficult to predict it exhaustively. Usually, this flight domain is investigated by means of systematic or Monte Carlo numerical simulations before the first flight and by means of extensive and expensive flight tests. Thanks to bifurcation theory and computer capabilities, a methodology and software have been set up to investigate asymptotic behaviour of nonlinear differential equations depending on parameters. This methodology is mainly used at ONERA to study the high-AOA behaviour of very realistic missile, aircraft and submarine models. It is also used in fields other than flight dynamics.

Coming back to the Alpha-Jet application, it can be seen that bifurcation theory has been used to identify an aerodynamic model suitable for the analysis of high-AOA flight regimes. To validate the methodology, flight tests were performed after prediction of aircraft behaviour by means of equilibrium surfaces and periodic-orbit envelopes for several aerodynamic formulations. Thanks to flight-test pilots, who were asked to perform rather unusual flight tests, very good correlation with results predicted by the theory has been obtained.

Thus, considering these results, it can be said that this technique has great potential and is appropriate for the investigation of aircraft behaviour, using only wind-tunnel data. However, one cannot forget that the quality of prediction is directly related to the quality of the aerodynamic database of the aircraft model.

As expected, another issue of the flight tests was that asymptotic behaviour was not always reached because of the duration of the tests. Furthermore, in many real problems, controls are not fixed or independent of time. It may follow transient motions and/or quasi-stationary behaviour which must be addressed. These problems are connected to the determination of the attracting basin of a stable equilibrium.

Much work has been done in this field. It is mainly based on Liapounov's stability theory and its extensions by Zubov and La Salle. More recently, introducing topological considerations, trajectory-reversing methods have been developed, and numerous computational procedures have been proposed. These procedures are appropriate to low-dimensional dynamic systems. However, there is a need to improve them for higher dimensions. Due to the difficulty of working with high-dimensional systems, a limited part of the attracting basin is generally computed. Is it sufficient? Are we interested in the entire domain of attraction? It seems that a lot of work is needed to give practical answers to this problem.

As a further step, the methodology can also be used to investigate nonlinear behaviour induced by nonlinear elements in flight-control systems. From a stability point of view and under implicit assumptions on the controlled system (continuous time, continuous nonlinearities, etc.) it has been shown that bifurcation theory can help the designer predict system behaviour and to compute 'good' control laws. Although promising results are available in the literature, some work remains in order to take into account practical systems. In this field, it seems also very interesting to complete the analysis by determining the region of asymptotic stability to quantify control-law robustness.

Finally, much work has been done on continuous systems. In practical applications, digital flight control and nonlinearities in control modify the behaviour of continuous systems. So, it becomes necessary to be able to numerically predict the behaviour of complex systems with a continuous part and a discrete-time part. This requires theoretical developments and modifications of numerical techniques which are of interest for ONERA.

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